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Reachability relations in digraphs

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Abstract

In this paper we study reachability relations on vertices of digraphs, informally defined as follows. First, the weight of a walk is equal to the number of edges traversed in the direction coinciding with their orientation, minus the number of edges traversed in the direction opposite to their orientation. Then, a vertex u is R_k^+-related to a vertex v if there exists a 0-weighted walk from u to v whose every subwalk starting at u has weight in the interval $[0, k]$. Similarly, a vertex u is R_k^--related to a vertex v if there exists a 0-weighted walk from u to v whose every subwalk starting at u has weight in the interval $[-k, 0]$. For all positive integers k, the relations R_k^+ and R_k^- are equivalence relations on the vertex set of a given digraph.

We prove that, for transitive digraphs, properties of these relations are closely related to other properties such as having property **Z**, the number of ends, growth conditions, and vertex degree.

1. Introduction

In [1], highly arc-transitive digraphs were considered from several different viewpoints, leading to – besides many nice results – a number of interesting problems. Some of them have already been completely or at least partially solved (see [4,9,10,12,13]). One problem which is still open is the following. Does there exist a connected locally finite highly arc-transitive digraph with universal reachability relation? Recall that, informally speaking, an edge f is *reachable* from an edge e if there exists an alternating walk such that the first edge we traverse by following

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this walk equals e and the last one equals f. Note that the notion of reachability between edges in a digraph without loops or pairs of vertices linked by two oppositely directed edges is one of the key concepts introduced in [1].

In this paper we study a family of reachability relations defined on the vertex set of a digraph (possibly with loops and pairs of vertices linked by two oppositely directed edges), which generalize a similar concept introduced in [11] in the context of finite simple digraphs. The informal definition is the following (for a precise definition see Section 2). First, the weight of a walk is defined to be the number of edges traversed in the direction coinciding with their orientation minus the number of edges traversed in the direction opposite to their orientation. Then, a vertex u is R_k^+-related to a vertex v if there exists a 0-weighted walk from u to v such that every subwalk starting at u has weight in the interval $[0, k]$. Similarly, a vertex u is R_k^--related to a vertex v if there exists a 0-weighted walk from u to v whose every subwalk starting at u has weight in the interval $[-k, 0]$. Note that R_k^+ and R_k^- are equivalence relations for all positive integers k.

We show that the interplay of certain properties of digraphs – such as having property $\mathbf{Z}$, number of ends, growth conditions, and vertex degree – is strongly related to the properties of the two sequences of equivalence relations $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$.

The paper is organized as follows. In Section 2 the reader will find formal definitions and notation. In Section 3 some general results concerning the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$, and particular relations R_k^+ and R_k^- are proved. Section 4 contains the main results of this paper. Among others we give two necessary and sufficient condition for a connected infinite locally finite transitive digraph to have property $\mathbf{Z}$ in terms of properties of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$, one for two-ended digraphs and one for digraphs with infinitely many ends (Theorem 4.5). We also prove that if at least one of these two sequences of equivalence relations is infinite, for some connected infinite locally finite transitive digraph, then the digraph in question must have exponential growth (Theorem 4.12).

2. Preliminaries

A *digraph* is an ordered pair $D = (V(D), E(D))$, where $V(D)$ is the vertex set and $E(D) \subseteq V(D) \times V(D)$ is the edge set. Note that a digraph can have loops (v, v) as well as pairs of 'oppositely directed' edges of the form (u, v) and (v, u). Digraphs considered in this paper are connected in the sense that their underlying undirected graphs are connected.

A *walk* $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ from v_0 to v_n of length $|W| = n \geq 0$ is a sequence of $n + 1$ (not necessarily pairwise distinct) vertices $v_0, v_1, \ldots, v_n \in V(D)$, and n indicators $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, -1\}$ such that for all $j \in \{1, 2, \ldots, n\}$ we have

$$\varepsilon_j = 1 \Rightarrow (v_{j-1}, v_j) \in E(D),$$
$$\varepsilon_j = -1 \Rightarrow (v_j, v_{j-1}) \in E(D).$$

W is called a *closed walk* if $v_0 = v_n$. Intuitively, a walk is a traversal in the digraph from vertex to vertex along edges, where indicators 1 and -1 tell whether the traversal respects the direction of edges or not. The formal definition of a walk as above has been chosen in order to make proofs unambiguous, and in order to facilitate the formal introduction of certain concepts to be defined later on.

If the vertices of a walk W are pairwise different then W is called a *path*. A walk (or a path) is *directed* if all indicators are equal to 1, and is *alternating* if the values of the indicators alternate.

Let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ be a walk. We let the *inverse walk* of W be $W^{-1} = (v_n, -\varepsilon_n, v_{n-1}, \ldots, -\varepsilon_1, v_0)$. Moreover, for $0 \le i \le j \le n$, the subsequence

$${}_iW_j = (v_i, \varepsilon_{i+1}, \ldots, \varepsilon_j, v_j)$$

of W is called a *subwalk*. Furthermore, let $W' = (u_0, \delta_1, u_1, \ldots, \delta_m, u_m)$ be a walk such that $u_0 = v_n$. Then the *concatenation* of W and W' is the walk

$$W \cdot W' = (v_0, \varepsilon_1, v_1, \ldots, v_{n-1}, \varepsilon_n, u_0, \delta_1, u_1, \ldots, \delta_m, u_m)$$

of length $n + m$.

We now introduce two families of reachability relations defined on vertices of a digraph. Let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ be a walk. The *weight* of the walk W is defined as

$$\zeta(W) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n.$$

Let $k \ge 1$. We say that a vertex $u \in V(D)$ is R_k^+*-related* to a vertex $v \in V(D)$, in symbols

$$u R_k^+ v,$$

if there exists a walk W from u to v such that $\zeta(W) = 0$, and that for every $0 \le j \le |W|$ we have $\zeta({}_0W_j) \in [0, k]$. For a given pair of vertices u, v, the set of all such walks is denoted by $R_k^+[u, v]$. Analogously we say that u is R_k^-*-related* to v, in symbols $u R_k^- v$, if there exists a walk W such that $\zeta(W) = 0$, and that for every $0 \le j \le |W|$ we have $\zeta({}_0W_j) \in [-k, 0]$. For a given pair of vertices u, v, the set of all such walks is denoted by $R_k^-[u, v]$. Note that R_k^+ and R_k^- are equivalence relations. Their equivalence classes are denoted by $R_k^+(v)$ and $R_k^-(v)$, $v \in V(D)$, respectively. If D is transitive, then the equivalence classes of R_k^+ (and similarly of R_k^-) form an imprimitivity system for $\mathrm{Aut}(D)$. Observe that the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ are ascending: for all k we have $R_k^+ \subseteq R_{k+1}^+$ and $R_k^- \subseteq R_{k+1}^-$. Their respective unions

$$R^+ = \bigcup_{k\in\mathbb{Z}^+} R_k^+ \quad \text{and} \quad R^- = \bigcup_{k\in\mathbb{Z}^+} R_k^-$$

are thus also equivalence relations, and their equivalence classes form imprimitivity systems for $\mathrm{Aut}(D)$ whenever D is transitive. Finally, we remark that if D has a loop at every vertex, then all the above relations are universal (recall that D is assumed to be connected).

For a vertex $v \in V(D)$ define $M_j^+(v)$ as the set of endvertices of all directed walks of length j and with origin v. Analogously, $M_j^-(v)$ denotes the set of origins of all directed walks of length j with endvertex v. If $S \subset V(D)$ is any subset of vertices of D, then $M_j^+(S) = \bigcup_{v\in S} M_j^+(v)$ and $M_j^-(S) = \bigcup_{v\in S} M_j^-(v)$. The *outdegree* of a vertex $v \in V(D)$ is the cardinality $d_D^+(v) = |M_1^+(v)|$ and the *indegree* of v is $d_D^-(v) = |M_1^-(v)|$. If outdegrees (or indegrees) are the same for all vertices, we write d_D^+ (or d_D^-); the subscript D is omitted if D is clear from the context.

Let $s \ge 0$ be an integer. An *s-arc* in a digraph D is a directed walk $(v_0, 1, v_1, \ldots, 1, v_s)$ of length s such that $v_j \ne v_{j+2}$ for all $0 \le j \le s - 2$. If the automorphism group $\mathrm{Aut}(D)$ of D acts transitively on the set of s-arcs, then D is called *s-arc-transitive*. If $s = 0$ we simply say that D is *transitive*. A digraph D is said to be *highly arc-transitive* if it is s-arc-transitive for all finite $s \ge 0$.

A *double-ray* $R = (\ldots, v_{-1}, v_0, v_1, \ldots)$ is a two-way infinite sequence of pairwise distinct vertices such that any two consecutive vertices are adjacent. The above double-ray is called a *line* if $(v_j, v_{j+1}) \in E(D)$ for every $j \in \mathbb{Z}$. A digraph D has *property* $\mathbf{Z}$ if there exists a homomorphism from D onto a directed infinite line which is a Cayley digraph of the additive group of integers. The subsequences $R[v_j, \infty) = (v_j, v_{j+1}, v_{j+2}, \ldots)$ and $R[v_j, -\infty) = (v_j, v_{j-1}, v_{j-2}, \ldots)$ of R are called *rays*. If L is a line, then $L[v_j, \infty)$ is a *positive halfline* and $L[v_j, -\infty)$ a *negative halfline*. If – in a certain context – the starting vertices of halflines are not important, then we write L^+ or L^- for positive and negative halflines.

Two rays P and Q in D are *equivalent* if in the underlying undirected graph there are infinitely many pairwise disjoint paths connecting vertices in P to vertices in Q. The equivalence classes of all rays with respect to this relation are called the *ends* of D. The concept of ends can be defined in several different ways; the above definition is due to Halin [6].

The *distance* $\mathrm{dist}_D(u, v)$ between vertices u and v in a connected digraph D is the length of a shortest path from u to v. The *growth function* $f_D(v, n)$, $n \geq 0$, with respect to some $v \in V(D)$ is given by

$$f_D(v, n) = |\{u \in V(D) \mid \mathrm{dist}_D(v, u) \leq n\}|.$$

If D is transitive then this function does not depend on a particular vertex $v \in V(D)$; in this case we denote it by $f_D(n)$. We say that a transitive digraph D has *exponential growth* if there is a constant $c > 1$ such that

$$f_D(n) > c^n$$

holds for all $n > 0$. We remark that this definition agrees with the usual definition in the context of undirected graphs.

3. Some properties of R_k^+ and R_k^-

In this section we consider relations R_k^+ and R_k^- from three different aspects. First we give some necessary and sufficient conditions for the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ to be finite. (The sequence $(R_k^+)_{k\in\mathbb{Z}^+}$, respectively $(R_k^-)_{k\in\mathbb{Z}^+}$, is *finite* if there exists $k \in \mathbb{Z}^+$ such that $R^+ = R_k^+$, respectively $R^- = R_k^-$.) Next, we study certain properties of equivalence classes of each individual relation R_k^+ and R_k^-. Finally, we consider quotients of digraphs with respect to an equivalence relation on vertices, in particular, quotients with respect to R_1^+ or R_1^-.

3.1. The sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$

As already remarked in Section 2, in a connected digraph with loops at all vertices the relations R_k^+ and R_k^- are all universal. For digraphs in which each vertex is contained in a directed closed walk of length 2, the situation is only slightly more involved.

Proposition 3.1. *Let D be a connected digraph in which each vertex is contained in a directed closed walk of length* 2. *Then $R^+ = R_2^+ = R_2^- = R^-$. Moreover,*

- *if there exists a closed walk of odd length in D, then R_2^+ (and thus R_2^-) is universal, and*
- *if there exists no closed walk of odd length in D, then R_2^+ (and thus R_2^-) has two equivalence classes.*

Proof. We first show that, given any two vertices u and v of D, an arbitrary walk $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ from u to v can be modified in such a way that we obtain a new walk

W' from u to v having the following properties: $\zeta({}_0W'_j) \in [0, 2]$ for all $j = 0, 1, \ldots, |W'|$ and $\zeta(W')$ is either 0 or 1 depending on whether $|W|$ is even or odd, respectively. The walk W' is defined recursively by traversing W with additional modification as the case may be. Initially we let W' be the trivial walk (u). Then, reaching the j-th vertex of W, where $0 \le j \le n$, we redefine W' depending on three possible cases as follows:

Case 1: $\zeta(W') = 0$ and $\varepsilon_{j+1} = 1$, or $\zeta(W') = 2$ and $\varepsilon_{j+1} = -1$, or $\zeta(W') = 1$ with ε_{j+1} arbitrary. In this case we redefine $W' = W' \cdot (v_j, \varepsilon_{j+1}, v_{j+1})$.

Case 2: $\zeta(W') = 0$ and $\varepsilon_{j+1} = -1$. Then there exists a vertex $w \in V(D)$ such that $(v_j, w), (w, v_j) \in E(D)$, and so we set $W' = W' \cdot (v_j, 1, w, 1, v_j, \varepsilon_{j+1}, v_{j+1})$.

Case 3: $\zeta(W') = 2$ and $\varepsilon_{j+1} = 1$. Then set $W' = W' \cdot (v_j, -1, w, -1, v_j, \varepsilon_{j+1}, v_{j+1})$, where w is as in Case 2.

Continuing this way we end up with a walk W' from u to v which clearly has all the required properties, except that possibly $\zeta(W') = 2$. If this is the case we redefine $W' = W' \cdot (v, -1, w, -1, v)$ for an appropriate vertex w.

It is now clear that R_2^+ has at most two equivalence classes as two vertices are R_2^+ related if and only if there exists a walk of even length from one to the other. That R_2^+ is universal if and only if there exists a closed walk of odd length in D is now also clear. Moreover, as the fact that uR^+v for some $u, v \in V(D)$ implies that there exists an appropriate walk of even length from u to v, we have $R^+ = R_2^+$.

The proof for R_2^- is done in a similar way. As the condition for two vertices $u, v \in V(D)$ to be R_2^+-related coincides with the condition for u and v to be R_2^--related, the fact that $R_2^+ = R_2^-$ follows. $\square$

Similar arguments show that when each vertex of a digraph is contained in a directed cycle of length k we have $R^+ = R_k^+ = R_k^- = R^-$. For a proof using different methods see [14]. Of course, one or both of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ can be finite even though the digraph in question has no directed cycles. However, as results of Section 4 show, knowing whether the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ or $(R_k^-)_{k\in\mathbb{Z}^+}$ are finite or infinite is of vital importance in studying properties of digraphs such as number of ends, property **Z**, growth, etc. We now give three necessary and sufficient conditions for the two sequences above to be finite. The first of them states, informally speaking, that these sequences are infinite if and only if they are 'strictly ascending'.

Proposition 3.2. *Let D be a digraph. Then $R^+ = R_k^+$ for some k if and only if $R_{k+1}^+ = R_k^+$. An analogous assertion holds for R^-.*

Proof. If $R^+ = R_k^+$, then clearly $R_{k+1}^+ = R_k^+$. For the converse, suppose that there exists a positive integer k such that $R_{k+1}^+ = R_k^+$. We show that then $R_{k+2}^+ \subseteq R_{k+1}^+$, which clearly implies $R_{k+2}^+ = R_{k+1}^+$, and consequently by induction $R^+ = R_k^+$ holds.

Consider two vertices u and v which are R_{k+2}^+-related, and let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n) \in R_{k+2}^+[u, v]$ be a walk establishing this relation. Let $0 < j < n$ be the smallest integer such that $\zeta({}_0W_j) = k + 2$. (If no such integer exists, the claim clearly holds.) Let $0 < j_1 < j$ be the largest integer such that $\zeta({}_0W_{j_1}) = 1$ and let $j < j_2 < n$ be the smallest integer such that $\zeta({}_{j_2}W_n) = 1$. Then ${}_{j_1}W_{j_2} \in R_{k+1}^+[v_{j_1}, v_{j_2}]$. By assumption there exists a walk $W' \in R_k^+[v_{j_1}, v_{j_2}]$. Let $W'' = {}_0W_{j_1} \cdot W' \cdot {}_{j_2}W_n$. If there still is some $0 < j' < |W''|$ such that $\zeta({}_0W''_{j'}) = k+2$, proceed as above. Continuing this way we end up with a walk which is an element of $R^+_{k+1}[u, v]$, as required.

The assertion for R^- can be shown analogously. □

Proposition 3.3. *Let D be a digraph with minimal indegree at least 1. Then for each integer $k \geq 1$ we have*

$$R^+ = R^+_k \iff R^+_{k+1} = R^+_k \iff R^+_1 \subseteq R^-_k \iff R^+ \subseteq R^-_k.$$

Exchanging the roles of $+$ and $-$ and substituting outdegree for indegree we obtain an analogous result.

Proof. Fix an integer $k \geq 1$. The first of the equivalencies is given by Proposition 3.2.

We now show that $R^+ = R^+_k$ implies $R^+ \subseteq R^-_k$. Let uR^+v (and thus uR^+_kv) and let $W \in R^+_k[u, v]$. Choose vertices u^* and v^* together with directed walks W^u and W^v of lengths k from u^* to u and from v^* to v, respectively. (Note that u^*, v^* and the respective directed walks exist, as D has minimal indegree at least 1.) Then the walk $W' = W^u \cdot W \cdot (W^v)^{-1}$ is clearly contained in $R^+_{2k}[u^*, v^*]$, and so some walk $W^* \in R^+_k[u^*, v^*]$ exists as $R^+ = R^+_k$. But then the walk $(W^u)^{-1} \cdot W^* \cdot W^v$ is contained in $R^-_k[u, v]$, as required.

That $R^+ \subseteq R^-_k$ implies $R^+_1 \subseteq R^-_k$ is clear, and so it remains to show that $R^+_1 \subseteq R^-_k$ implies $R^+_{k+1} = R^+_k$. To this end let $uR^+_{k+1}v$. Moreover, let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n) \in R^+_{k+1}[u, v]$ and let $0 < j < n$ be the smallest integer such that $\zeta({}_0W_j) = k+1$. (If no such j exists, there is nothing to prove.) Then, $\zeta({}_0W_{j-1}) = \zeta({}_0W_{j+1}) = k$, and so $v_{j-1}R^+_1v_{j+1}$. By assumption some $W' \in R^-_k[v_{j-1}, v_{j+1}]$ exists. Set $W'' = {}_0W_{j-1} \cdot W' \cdot {}_{j+1}W_n$. If there still is some $0 < j' < |W''|$ such that $\zeta({}_0W''_{j'}) = k+1$, repeat the above procedure. Continuing this way we end up with a walk which is contained in $R^+_k[u, v]$, as required.

The proof of the 'dual' version is done in an analogous way. □

Corollary 3.4. *Let D be a digraph. Suppose that $R^+_k = R^-_k$ for some integer $k \geq 1$. Then $R^+ = R^+_k = R^-_k = R^-$.*

Proof. By definition $R^+_1 \subseteq R^+_k$, and so $R^+_1 \subseteq R^-_k$. By the arguments given in the third paragraph of the proof of Proposition 3.3 we have $R^+_{k+1} = R^+_k$. (Note that this part of the proof of Proposition 3.3 does not require that D has minimal indegree at least 1.) Hence $R^+ = R^+_k$, by Proposition 3.2. That $R^- = R^-_k$ also holds is proved in a similar manner. □

Corollary 3.5. *Let D be a digraph with minimal indegree and outdegree at least 1. Then the following are equivalent:*

(i) *The sequences $(R^+_k)_{k\in\mathbb{Z}^+}$ and $(R^-_k)_{k\in\mathbb{Z}^+}$ are both finite.*
(ii) *There exists some integer $k \geq 1$ such that $R^+ = R^+_k = R^-_k = R^-$ with $R^+_k \neq R^+_{k-1}$, $R^-_k \neq R^-_{k-1}$ and $R^+_{k-1} \neq R^-_{k-1}$ whenever $k \geq 2$.*

Proof. Suppose first that both of the sequences are finite, that is, there exist minimal integers $i, j \geq 1$ such that $R^+ = R^+_i$ and $R^- = R^-_j$. Set $k = \max(i, j)$. Then $R^+ = R^+_k$ and $R^- = R^-_k$, and so Proposition 3.3 implies that $R^+ \subseteq R^-_k = R^-$ and $R^- \subseteq R^+_k = R^+$. Thus $R^+ = R^+_k = R^-_k = R^-$. Suppose now that $k \geq 2$, and in addition, suppose without loss of generality that $R_k^+ \neq R_{k-1}^+$ and $R_k^- = R_{k-1}^-$ (that is, $j < i$). Then $R^+ = R_k^+ = R_k^- = R_{k-1}^-$, and so Proposition 3.3 implies that $R^+ = R_{k-1}^+$. But this contradicts the fact that $R_k^+ \neq R_{k-1}^+$. We have thus shown that $R_k^+ \neq R_{k-1}^+$ and $R_k^- \neq R_{k-1}^-$. Since $R_k^+ \neq R_{k-1}^+$, and $k \geq 2$, Corollary 3.4 implies that $R_{k-1}^+ \neq R_{k-1}^-$.

The converse is trivial. □

3.2. The equivalence classes of R_k^+ and R_k^-

Throughout this subsection we consider digraphs D with minimal indegree and minimal outdegree at least 1. Observe that under these assumptions $M_k^+(v)$ and $M_k^-(v)$ are nonempty for all vertices $v \in V(D)$ and all integers $k \geq 1$, that is, directed walks of arbitrarily large length exist in D. We prove results regarding the relation between equivalence classes of R_k^+ and R_k^- and their cardinalities.

Proposition 3.6. *Let D be a digraph with minimal indegree and minimal outdegree at least 1, let $v \in V(D)$ and let $k \geq 1$. Then for each $u \in M_k^+(R_k^+(v))$ we have $M_k^+(R_k^+(v)) = R_k^-(u)$. Exchanging the roles of $+$ and $-$ we obtain an analogous result.*

Proof. As $u \in M_k^+(R_k^+(v))$, there exists a vertex $v' \in R_k^+(v)$ and a directed walk $W^{v'}$ of length k from v' to u.

We first show that $M_k^+(R_k^+(v)) \subseteq R_k^-(u)$. Let $u' \in M_k^+(R_k^+(v))$. Thus, there exists some $v'' \in R_k^+(v)$ and some directed walk $W^{v''}$ of length k from v'' to u'. Choose a walk $W' \in R_k^+[v', v]$ and a walk $W'' \in R_k^+[v'', v]$. Then the walk $W = (W^{v''})^{-1} \cdot W'' \cdot (W')^{-1} \cdot W^{v'}$ is clearly contained in $R_k^-[u', u]$, as required.

To show that $R_k^-(u) \subseteq M_k^+(R_k^+(v))$ let $u' \in R_k^-(u)$. Choose $W' \in R_k^-[u, u']$. Since the minimal indegree is at least 1, there exists a vertex $v'' \in V(D)$ and a directed walk $W^{v''}$ of length k from v'' to u'. Then the walk $W = W^{v'} \cdot W' \cdot (W^{v''})^{-1}$ is clearly contained in $R_k^+[v', v'']$. Thus $v'' \in R_k^+(v)$, and so $u' \in M_k^+(R_k^+(v))$, as required.

The 'dual' version can be proved analogously. □

Proposition 3.7. *Let D be a connected locally finite transitive digraph and let $k \geq 1$ be an integer. Suppose that, for some (and hence any) $v \in V(D)$, at least one of the equivalence classes $R_k^+(v)$ and $R_k^-(v)$ is finite. Then $R_k^+(v)$ and $R_k^-(v)$ are both finite. In fact,*

$$\frac{|R_k^+(v)|}{|R_k^-(v)|} = \left(\frac{d^-}{d^+}\right)^k. \tag{1}$$

Proof. Clearly $d^+, d^- \geq 1$, and so Proposition 3.6 applies. Suppose, without loss of generality, that $R_k^+(v)$ is finite. Choose a directed walk of length k starting at v, and let $u \in M_k^+(v)$ be its terminal vertex. By Proposition 3.6 we have that $M_k^+(R_k^+(v)) = R_k^-(u)$. Since D is locally finite and the set $R_k^+(v)$ is finite, $R_k^-(u)$ is finite as well. By transitivity, $R_k^-(v)$ is finite.

To obtain the equality (1), note that Proposition 3.6 also implies that $M_k^-(R_k^-(u)) = R_k^+(v)$ holds. Together with $M_k^+(R_k^+(v)) = R_k^-(u)$ this implies that every directed walk of length k starting in $R_k^+(v)$ ends in $R_k^-(u)$ and conversely, that every directed walk of length k ending in $R_k^-(u)$ starts in $R_k^+(v)$. Counting the number of directed walks of length k which start in $R_k^+(v)$ and end in $R_k^-(u)$ in two different ways we thus have that $|R_k^+(v)|(d^+)^k = |R_k^-(v)|(d^-)^k$, which yields Eq. (1). □

Corollary 3.8. *Let D be a connected locally finite transitive digraph and let $k \geq 1$ be an integer. Suppose that for some (and hence any) $v \in V(D)$, at least one of the equivalence classes $R_k^+(v)$ and $R_k^-(v)$ is finite. Then $|R_k^+(v)| = |R_k^-(v)|$ if and only if $d^+ = d^-$.* □

Lemma 3.9. *Let D be a connected transitive digraph, let $v \in V(D)$, and let $k \geq 1$ be an integer. Then for any $u \in M_k^+(v)$ and for any $g \in \mathrm{Aut}(D)$ we have that $g(R_k^+(v)) = R_k^+(v)$ if and only if $g(R_k^-(u)) = R_k^-(u)$.*

Proof. Suppose that $g(R_k^+(v)) = R_k^+(v)$. Let W^v be a directed walk of length k, starting at v and ending at u, and let $W \in R_k^+[v, g(v)]$. Then $(W^v)^{-1} \cdot W \cdot g(W^v) \in R_k^-[u, g(u)]$, and so $g(R_k^-(u)) = R_k^-(u)$, as $R_k^-(u)$ is a block of imprimitivity for $\mathrm{Aut}(D)$. The converse implication can be shown analogously. □

We end this subsection by giving a necessary condition for a connected transitive digraph to be a Cayley digraph.

Proposition 3.10. *Let D be a connected Cayley digraph. Then for all integers $k \geq 1$ and for all $v \in V(D)$ we have that $|R_k^+(v)| = |R_k^-(v)|$.*

Proof. If D is locally finite then the claim follows by Corollary 3.8. Hence we can assume that D is locally infinite (even though the proof below works for all digraphs).

Let $k \geq 1$ and let $v \in V(D)$. Choose $u \in M_k^+(v)$ and let G be a subgroup of automorphisms of D acting regularly on its vertices. Such a subgroup G exists since D is a Cayley digraph. By Lemma 3.9 the block $R_k^+(v)$ is invariant for an automorphism $g \in G$ if and only if the block $R_k^-(u)$ is invariant for g. As G is regular, the cardinality of the set of all such $g \in G$ is equal to $|R_k^+(v)|$ and at the same time to $|R_k^-(u)|$. Since D is transitive, the result follows. □

3.3. *Quotients*

We now consider quotients of a digraph with respect to an equivalence relation on its vertices. Let D be a digraph and let τ denote the partition of $V(D)$ given by some equivalence relation on $V(D)$. The equivalence class of a vertex $v \in V(D)$ is denoted by v^τ. Then the *quotient digraph with respect to* τ, denoted by D^τ, is the digraph with vertex set τ with $(u^\tau, v^\tau) \in E(D^\tau)$ if and only if there exist $u' \in u^\tau$ and $v' \in v^\tau$ such that $(u, v) \in E(D)$. If $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ is a walk in D, then the *quotient walk* W^τ of W is defined to be the walk

$$W^\tau = (v_0^\tau, \varepsilon_1, v_1^\tau, \ldots, \varepsilon_n, v_n^\tau).$$

We emphasize that for every $0 \leq j \leq |W| = |W^\tau|$ we have $\zeta({}_0W_j) = \zeta({}_0W_j^\tau)$.

We now show that R_k^+ or R_k^- on D is closely related to R_k^+ or R_k^- on D^τ, respectively, where τ is the partition given by R_1^+ or R_1^-, respectively. Additional results on quotients of digraphs with respect to relations R_k^+ and R_k^- can be found in [14].

Proposition 3.11. *Let D be a digraph, let τ be the partition of $V(D)$ given by R_1^+, and let $u \in V(D)$. Then, for any $v \in V(D)$ and any $k \geq 2$ we have that uR_k^+v if and only if $u^\tau R_{k-1}^+ v^\tau$. An analogous assertion holds for R_k^- when taking the quotient with respect to R_1^-.*

Proof. Let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n) \in R_k^+[u, v]$, and let $0 < j < n$ be the smallest integer such that $\zeta({}_0W_j) = k$. (If no such j exists, then the quotient walk W^τ is in

$R^+_{k-1}[u^\tau, v^\tau]$.) Then ${}_{j-1}W_{j+1} \in R^+_1[v_{j-1}, v_{j+1}]$. So $v^\tau_{j-1} = v^\tau_{j+1}$, and we can replace W^τ by $W' = {}_0W^\tau_{j-1} \cdot {}_{j+1}W^\tau_n$. If there still is some $0 < j' < |W'|$ such that $\zeta({}_0W'_{j'}) = k$, proceed as above. Continuing this way we end up with a walk which is an element of $R^+_{k-1}[u^\tau, v^\tau]$, as required.

To prove the converse, let $\bar{W} = (u^\tau_0, \varepsilon_1, u^\tau_1, \ldots, \varepsilon_n, u^\tau_n)$ be such that $u \in u^\tau_0$, $v \in u^\tau_n$, and that $\bar{W} \in R^+_{k-1}[u^\tau_0, u^\tau_n]$. For each $1 \le j \le n$ there exist vertices $u'_{j-1} \in u^\tau_{j-1}$ and $u''_j \in u^\tau_j$ such that $(u'_{j-1}, \varepsilon_j, u''_j)$ is a walk in D and $(u^\tau_{j-1}, \varepsilon_j, u^\tau_j)$ is a walk in the quotient digraph. Set $u''_0 = u$ and $u'_n = v$. For each $0 \le j \le n$ let $W^j \in R^+_1[u''_j, u'_j]$. Then the walk $W = W^0 \cdot (u'_0, \varepsilon_1, u''_1) \cdot W^1 \cdots W^{n-1} \cdot (u'_{n-1}, \varepsilon_n, u''_n) \cdot W^n$ is clearly an element of $R^+_k[u, v]$, as required.

The assertion for R^-_k can be shown analogously. □

4. Reachability relations and property Z

Throughout this section D will always denote a connected infinite transitive digraph, unless explicitly stated otherwise. This section is split into three subsections. The first gives some sufficient conditions for D to have property **Z**. The second studies the connections between properties of the reachability relations R^+_k and R^-_k, property **Z**, and the number of ends of D. In the last subsection we prove a theorem connecting the growth of D and properties of R^+_k and R^-_k.

4.1. Conditions for property Z

Observe that D has property **Z** if and only if all closed walks in D are balanced. (A walk W in D is *balanced* if $\zeta(W) = 0$, and is *nonbalanced* otherwise.) In fact, it is enough to require that all simple closed walks in D are balanced, as the reader can easily check. (A closed walk W of length $n \ge 1$ is *simple* if the subwalk ${}_0W_{n-1}$ is a path.) The next technical lemma gives another necessary and sufficient condition for D to have property **Z** in terms of closed walks.

Lemma 4.1. *Let D be a connected infinite transitive digraph. Then D does not have property* **Z** *if and only if there exist integers $1 \le s \le t$ such that for each $v \in V(D)$ there exists a simple closed walk W^v at v with $\zeta(W^v) = s$ and $\zeta({}_0W^v_j) \in [0, t]$ for all $0 \le j \le |W^v|$.*

Proof. Suppose that D does not have property **Z**. By the remark at the beginning of this subsection, nonbalanced simple closed walks exist in D. Let $C = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$, where $v_n = v_0$, be a nonbalanced simple closed walk of shortest length. As $\zeta(C) \ne 0$ we may without loss of generality assume that $s = \zeta(C) > 0$ (otherwise take C^{-1}). Let $0 \le i \le n$ be an integer for which $\zeta({}_0C_i)$ is minimal and set $u = v_i$. Then the walk $W^u = {}_iC_n \cdot {}_0C_i$, which is just a cyclic rotation of C, is such that $\zeta({}_0W^u_j) \ge 0$ for all j. Let $t = \max\{\zeta({}_0W^u_j) \mid 0 \le j \le n\}$ be the maximum of these weights. Clearly, $\zeta({}_0W^u_j) \in [0, t]$ for all $0 \le j \le n$ and $\zeta(W^u) = s$. By transitivity a closed walk W^w with these properties (for the same s and t) exists for each $w \in V(D)$.

The converse is obvious. □

Using the above lemma we now give two sufficient conditions in terms of properties of relations R^+_k and R^-_k for D to have property **Z**. We start by remarking that directed paths of arbitrarily large length exist in D. This holds by a result of Trofimov [16] when D is locally finite, and is straightforward when D is locally infinite. (Indeed, since at least one of d^+ and d^- is infinite, such a path can be constructed inductively because the number of vertices already visited is always finite, and the number of choices for a new vertex at each step is infinite.)

Proposition 4.2. *Let D be a connected infinite locally finite transitive digraph. Suppose that for each integer $k \geq 1$ at least one (and hence both) of the relations R_k^+ and R_k^- has finite equivalence classes. Then D has property* **Z**.

Proof. Suppose on the contrary that D does not have property **Z**. Let $1 \leq s \leq t$ be the integers given by Lemma 4.1 and set $k = t$. Choose $v \in V(D)$, let $m = |R_k^+(v)|$ (which is finite by assumption and Proposition 3.7), and let $P = (v_0, 1, v_1, \ldots, 1, v_{ms})$ be any directed path of length ms starting at v. Then the vertices $v_0, v_s, \ldots, v_{ms}$ are all R_k^+-equivalent. Namely, letting W^w denote the nonbalanced closed walk at $w \in V(D)$ given by Lemma 4.1, we have $W^{v_s} \cdot ({}_0P_s)^{-1} \in R_k^+[v_s, v_0]$, $W^{v_{2s}} \cdot ({}_sP_{2s})^{-1} \in R_k^+[v_{2s}, v_s]$, etc. Hence $m = |R_k^+(v)| \geq m + 1$, a contradiction which completes the proof. □

The next proposition gives a sufficient condition for D to have property **Z** in terms of sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$. This result is also proved in [14]. We present an alternative proof.

Proposition 4.3. *Let D be a connected infinite transitive digraph. If at least one of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite, then D has property* **Z**.

Proof. Suppose on the contrary that D does not have property **Z**. Let $1 \leq s \leq t$ be the integers given by Lemma 4.1 and let W^w denote the corresponding nonbalanced closed walk at $w \in V(D)$. Set $k = t$.

We now show that $R_{k+2}^+ = R_{k+1}^+$, which by Proposition 3.2 implies that $R^+ = R_{k+1}^+$. To this end let uR_{k+2}^+v for some $u, v \in V(D)$ and let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n) \in R_{k+2}^+[u, v]$. We construct a walk $W' \in R_{k+1}^+[u, v]$ by making suitable adjustments to W recursively as follows. Let $0 < i < n$ be the smallest integer such that $\zeta({}_0W_i) = k+2$ and let $i < j \leq n$ be the smallest integer such that $\zeta({}_0W_j) = s - 1$ (if there is no such i there is nothing to prove, and if such an i exists then j also exists since $1 \leq s \leq k$). We let W' be the walk

$$ {}_0W_{i-1} \cdot (W^{v_{i-1}})^{-1} \cdot {}_{i-1}W_{j-1} \cdot W^{v_{j-1}} \cdot {}_{j-1}W_n . $$

The reader may check that $W' \in R_{k+2}^+[u, v]$. It is clear that the number of integers $0 < i' < |W'|$ for which $\zeta({}_0W'_{i'}) = k+2$ is strictly smaller than the number of integers $0 < i < |W|$ for which $\zeta({}_0W_i) = k + 2$. Continuing eliminating the 'peaks' we thus finally obtain a walk from u to v which belongs to $R_{k+1}^+[u, v]$, as required.

A similar proof shows that $R^- = R_{k+1}^-$. This completes the proof. □

4.2. The number of ends

In this subsection we combine the results proved so far in order to better understand, in terms of relations R_k^+ and R_k^-, the interplay between property **Z** and the number of ends of a connected infinite locally finite transitive digraph.

Theorem 4.4. *Let D be a connected infinite locally finite transitive digraph with property* **Z**. *If D has more than two ends, then at least one of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite.*

Proof. Suppose on the contrary that the sequences are both finite. Then Corollary 3.5 implies that there exists an integer $k \geq 1$ such that $R^+ = R_k^+ = R_k^- = R^-$. Let $F_j = \varphi^{-1}(j)$, $j \in \mathbb{Z}$, denote the fibres of some epimorphism $\varphi : D \to$ **Z**.

Observe first that two vertices $u, v \in V(D)$ are R_k^+-related if and only if they belong to the same fibre F_j. Namely, if $u, v \in F_j$, then let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ be a walk from u to v and let $0 = i_1 < i_2 < \cdots < i_m = n$ be all the integers for which $v_{i_h} \in F_j$. Then each subwalk ${}_{i_h}W_{i_{h+1}}$ belongs either to $R^+[v_{i_h}, v_{i_{h+1}}]$ or to $R^-[v_{i_h}, v_{i_{h+1}}]$, and so the claim holds as $R_k^+ = R^+ = R^-$. The converse is obvious.

As D has property $\mathbf{Z}$, there exists a line $L = (\ldots, v_{-1}, v_0, v_1, v_2, \ldots)$ in D. We now show that an arbitrary ray Q of D is equivalent either to the ray $L^+ = L[v_0, \infty)$ or to the ray $L^- = L[v_0, -\infty)$, which contradicts the assumption that D has more than two ends.

Suppose first that Q intersects infinitely many fibres F_j. Then there exists some integer m such that either Q intersects F_n for every $n \geq m$, or Q intersects F_n for every $n \leq m$. We show that in the former case Q is equivalent to L^+ (the proof that in the latter case Q is equivalent to L^- is done analogously). With no loss of generality assume that $m = 0$. For each integer $j \geq 0$ choose a vertex $u_j \in Q \cap F_{j(k+1)}$. Then for each j let $W^j \in R_k^+[u_j, v_{j(k+1)}]$ (which exists by the above observation). As the vertices of W^j are contained in the union $F_{j(k+1)} \cup F_{j(k+1)+1} \cup \cdots \cup F_{j(k+1)+k}$ we have obtained infinitely many pairwise disjoint walks connecting Q to L^+. Hence we can also find appropriate paths, as required.

Suppose now that Q intersects only finitely many fibres. Thus Q intersects some F_j in infinitely many vertices. Without loss of generality we can assume that $Q \subset \cup_{j=0}^m F_j$ for some positive integer m, and moreover, that $S = Q \cap F_0$ is infinite. Denote $S = \{u_\lambda \mid \lambda \in \Lambda\}$. Since D is transitive, we have that for each $\lambda \in \Lambda$ there exists a positive halfline L_λ starting at u_λ. Recall that vertices $u, v \in V(D)$ are R_k^+ related if and only if they belong to the same fibre F_j. Let $\lambda_0 \in \Lambda$. Then $v_0 R_k^+ u_{\lambda_0}$, and so we can choose some $W^{\lambda_0} \in R_k^+[v_0, u_{\lambda_0}]$. Let T denote the set of all the vertices of W_{λ_0}. The facts that D is locally finite, that T is finite, and that Λ is infinite, imply that there is some $\lambda_1 \in \Lambda$ such that $L_{\lambda_1} \cap T = \emptyset$. We now construct a walk W^{λ_1} from u_{λ_1} to v_{k+1} as follows. We first traverse L_{λ_1} until we meet a vertex w_{k+1} of F_{k+1}. We then traverse a walk from $R_k^+[w_{k+1}, v_{k+1}]$. Clearly W^{λ_1} does not meet the vertices of T. Redefine T as the union of T and the vertices of W^{λ_1}. Continuing this way we obtain arbitrarily many pairwise disjoint walks connecting L^+ to Q. Hence Q is equivalent to L^+, which completes the proof. □

Note that the claim of Theorem 4.4 is best possible in the sense that transitive digraphs with property $\mathbf{Z}$ and infinitely many ends, and such that exactly one of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite, exist. The most simple examples are trees with indegree equal to one and outdegree equal to two (or vice versa). The lexicographic products of such trees with the digraph nK_1 (the digraph with n vertices and no edges), where $n \geq 2$, gives rise to examples where $R^+ = R_1^+$ (with nontrivial equivalence classes) and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite (or $R^- = R_1^-$ and $(R_k^+)_{k\in\mathbb{Z}^+}$ is infinite).

Combining the results of Section 4.1 and the above theorem we obtain the following result and its corollaries.

Theorem 4.5. *Let D be a connected infinite locally finite transitive digraph. Then the following hold.*

(i) *If D has two ends, then it has property $\mathbf{Z}$ if and only if for each integer $k \geq 1$ at least one (and hence both) of the relations R_k^+ and R_k^- have finite equivalence classes.*
(ii) *If D has infinitely many ends, then it has property $\mathbf{Z}$ if and only if at least one of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite.*

Proof. To prove (i) note that Propositions 3.7 and 4.2 imply that we only need to show that in a two-ended digraph with property $\mathbf{Z}$ every R_k^+ has finite equivalence classes. Since such a digraph must have finite fibres with respect to any epimorphism $D \to \mathbf{Z}$, see for instance [7], the claim follows trivially as each equivalence class of any R_k^+ is completely contained in some fibre.

As for claim (ii), it follows from Proposition 4.3 and Theorem 4.4.

Corollary 4.6. *Let D be a connected infinite locally finite transitive digraph with property $\mathbf{Z}$. If the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ are both finite and there exists an integer $k \geq 1$ such that the relation R_k^+ (and hence R_k^-) has infinite equivalence classes, then D has one end.* □

Note that digraphs as in Corollary 4.6 exist. A trivial example is the well known grid-digraph, that is, the Cayley digraph $\mathrm{Cay}(\mathbb{Z} \times \mathbb{Z}, \{(1, 0), (0, 1)\})$. Clearly, here $R_1^+ = R^+ = R^- = R_1^-$ which has equivalence classes of infinite cardinality. In view of this example no necessary and sufficient condition for property $\mathbf{Z}$ in one-ended transitive digraphs can be obtained just by combining the conditions given in Propositions 4.2 and 4.3. We therefore propose the following problem.

Problem 4.7. Find a necessary and sufficient condition in terms of relations R_k^+ and R_k^- for a connected infinite locally finite transitive one-ended digraph to have property $\mathbf{Z}$.

Still, quite a lot of information regarding the number of ends of a digraph can be extracted just by checking whether the two conditions given in Propositions 4.2 and 4.3 are satisfied or not.

Corollary 4.8. *Let D be a connected infinite locally finite transitive digraph. If all the relations R_k^+ (and hence R_k^-) have finite equivalence classes and the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ are both finite, then D has two ends.*

Proof. By Corollary 3.5 we have that $R^+ = R_k^+ = R_k^- = R^-$ for some $k \geq 1$. Moreover, Proposition 4.2 implies that D has property $\mathbf{Z}$. Thus, the proof of Theorem 4.4 shows that two vertices of D are R_k^+-related if and only if they belong to the same fibre of an epimorphism onto $\mathbf{Z}$. It follows that the fibres are finite, and so D has at least two ends. On the other hand Theorem 4.4 implies that D has at most two ends, which completes the proof. □

Using Theorem 4.5 and Corollaries 4.6 and 4.8 we find that the number of ends in a connected infinite locally finite transitive digraph with property $\mathbf{Z}$ is almost uniquely determined by the two conditions mentioned above. The information is gathered in Table 1, where for each possibility for the two conditions the possible number of ends is given. In the table the following shorthand notation is used for the two conditions:

$$\mathrm{Cond}_{\mathrm{cls}}: \text{ all } R_k^+ \text{(and hence } R_k^-\text{) have finite equivalence classes}$$

$$\mathrm{Cond}_{\mathrm{seq}}: \text{ at least one of } (R_k^+)_{k\in\mathbb{Z}^+} \text{ and } (R_k^-)_{k\in\mathbb{Z}^+} \text{ is infinite.}$$

Table 1
Transitive digraphs with property $\mathbf{Z}$ and the number of ends

$\neg\mathrm{Cond}_{\mathrm{cls}}, \neg\mathrm{Cond}_{\mathrm{seq}}$	$\mathrm{Cond}_{\mathrm{cls}}, \neg\mathrm{Cond}_{\mathrm{seq}}$	$\neg\mathrm{Cond}_{\mathrm{cls}}, \mathrm{Cond}_{\mathrm{seq}}$	$\mathrm{Cond}_{\mathrm{cls}}, \mathrm{Cond}_{\mathrm{seq}}$
1 end	2 ends	1 or ∞ ends	1 or ∞ ends

It is natural to ask whether all possibilities given in Table 1 actually occur. The answer is positive. We have already seen that digraphs corresponding to the first column do exist (the grid-digraph). That digraphs corresponding to the second column exist as well is also clear (for instance the **Z** digraph). Moreover, there are digraphs with infinitely many ends corresponding to the third column, and there are digraphs with infinitely many ends corresponding to the fourth column; take for instance the regular tree with indegree and outdegree 2 and the regular tree with indegree 1 and outdegree 2, respectively. Finally, Example 4.9 below shows that there are one-ended digraphs corresponding to the third column, and Example 4.10 shows that one-ended digraphs satisfying the assumptions of the fourth column exist too.

Example 4.9. Let T be an infinite directed tree with indegree 1 and outdegree 2. Let $\bar{T}$ be the multidigraph obtained from T by replacing each edge by a pair of parallel edges (both having the same direction as the original edge). Finally, let D be the $\mathbb{Z}$-cover of $\bar{T}$ arising from the ordinary voltage assignment on $\bar{T}$, where in each pair of parallel edges, one edge carries the voltage 0 and the other carries the voltage 1. (For a reference on covering graphs and voltage assignments see for e. g. [5,8].) For $\bar{u} \in V(\bar{T})$ let the *fibre* $\mathrm{Fib}(\bar{u})$ be the set of vertices of D which project to $\bar{u}$. We now show that D is a connected infinite locally finite one-ended transitive digraph with property **Z** which falls into the third column of Table 1.

That D is a connected infinite locally finite digraph with property **Z** is clear. Also, $R^+ = R_1^+$ (the equivalence classes of any R_k^+ are precisely the fibres of the projection onto $\bar{T}$) while the sequence $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite.

We now show that the digraph D is transitive. Indeed, let $g_{\bar{u}\bar{v}}$ be an automorphism of $\bar{T}$ mapping the vertex $\bar{u}$ to the vertex $\bar{v}$, and sending each edge in $\bar{T}$ with voltage 0 to an edge with voltage 0 (and hence each edge with voltage 1 to an edge with voltage 1). Such an automorphism clearly exists for any pair of vertices $\bar{u}$ and $\bar{v}$ of $\bar{T}$. Observe that $g_{\bar{u}\bar{v}}$ maps any closed walk with net voltage 0 to a closed walk with net voltage 0. By the basic lifting lemma [8], an automorphism lifts along a regular covering projection if and only if the set of closed walks with trivial net voltage is invariant under the action of this automorphism. Hence $g_{\bar{u}\bar{v}}$ lifts to an automorphism $\tilde{g}_{\bar{u}\bar{v}}$ of D mapping the fibre $\mathrm{Fib}(\bar{u})$ to the fibre $\mathrm{Fib}(\bar{v})$. As the group of covering transformations acts transitively on each fibre, the digraph D is transitive.

Finally, we show that D has one end. Let Q_1 and Q_2 be two rays in D, and let $\bar{Q}_1$ and $\bar{Q}_2$ be their respective projections in $\bar{T}$. Suppose $\bar{Q}_1$ contains finitely many, whereas $\bar{Q}_2$ contains infinitely many vertices. Then $\bar{Q}_2$ contains a ray, say $\bar{Q} = (\bar{v}_1, \bar{v}_2, \ldots) \subset \bar{Q}_2$, as a subsequence. For each $\bar{v}_j$ choose a vertex $v_j \in \mathrm{Fib}(\bar{v}_j) \cap Q_2$. There exists a vertex of $\bar{Q}_1$, say $\bar{u}$, such that the set $U = \mathrm{Fib}(\bar{u}) \cap Q_1$ is infinite. We are now going to construct infinitely many pairwise disjoint walks in D connecting suitably chosen vertices of U with vertices of $\{v_j \mid j \geq 1\}$, one for each j. Let $\bar{W}^1$ be the unique walk of $\bar{T}$ from $\bar{u}$ to $\bar{v}_1$ such that its corresponding edges all have voltage 0. Without loss of generality we may assume that the only common vertex of $\bar{W}^1$ and $\bar{Q}$ is $\bar{v}_1$. Construct the walks $\bar{W}^j$, $j \geq 2$, recursively by concatenating the walk $\bar{W}^{j-1}$ with the walk of length 1 from $\bar{v}_{j-1}$ to $\bar{v}_j$ having voltage 0. Choose a vertex $u_1 \in U$. Let $\tilde{W}^1$ be the lift of $\bar{W}^1$ starting at u_1. Next, define the walks $\tilde{W}^j$, $j \geq 2$, as follows. For each $j \geq 2$ choose a vertex $u_j \in U \setminus \{u_1, \ldots, u_{j-1}\}$ such that the lift $\tilde{W}^j$ of $\bar{W}^j$ starting at u_j avoids all vertices $v_1, v_2, \ldots, v_{j-1}$. Since U is infinite such vertices u_j exist. Note that the walks $\tilde{W}^j$, $j \geq 1$, are pairwise disjoint. Finally, for each $j \geq 1$ let W^j be the walk $\tilde{W}^j \cdot \hat{W}^j$, where $\hat{W}^j$ is a certain walk (which might be trivial) from the terminal vertex of $\tilde{W}^j$ to v_j. The walk $\hat{W}^j$ is chosen in such a way that its projection in $\bar{T}$ avoids all vertices of $\bar{Q}$ and all vertices of $\bar{W}^j$, except for $\bar{v}_j$, (the reader may check that such a walk exists). This way infinitely many pairwise disjoint walks W^j connecting vertices of Q_1 to vertices of Q_2 are obtained. Hence any ray of D whose projection contains finitely many vertices is equivalent to any ray of D whose projection contains infinitely many vertices. Since rays of both of these two types clearly exist in D, and since the relation of being equivalent is transitive, any two rays of D are equivalent. Hence D has one end. △

Example 4.10. The following family of digraphs, known as the 'broom digraphs', was introduced by Möller [12] to show that there exist infinite locally finite highly arc-transitive digraphs with one end. Undirected versions of these digraphs were independently considered by Woess [17] in the context of random walks, and by Diestel and Leader [2] in the context of quasi-isometry. Recently Eskin, Fisher and Whyte proved [3] that the graphs constructed by Diestel and Leader are in fact examples of graphs which are not quasi-isometric to any Cayley graph. To keep the paper self-contained we now give an informal description of the 'broom digraphs', following [12].

Let T be an infinite directed tree with indegree equal to one and outdegree equal to $n \geq 2$. Let $L = (\ldots, v_{-1}, v_0, v_1, \ldots)$ be a line in T. For each $j \in \mathbb{Z}$ let $M^-(v_j)$ denote the set $\cup_{i=0}^{\infty} M_i^-(R^-(v_j))$ and let G_j be the subdigraph of T induced on $M^-(v_j)$. Note that the digraphs G_j are pairwise isomorphic. We also remark that in [12] the set $R^-(v_j)$ was referred to as a *horocycle*. For each $j \in \mathbb{Z}$ we now glue a copy of the digraph $G \cong G_j$ onto the tree T by identifying the vertices of the horocycle $R^-(v_j)$ in a natural way. Denote the resulting digraph by D_1. Observe that the vertices of D_1 have outdegree n, and that the vertices corresponding to vertices of T have indegree equal to 2 while those which do not correspond to vertices of T have indegree equal to 1. Next, at each horocycle containing the vertices of D_1 with indegree 1 glue another copy of G in a natural way in order to obtain the digraph D_2. Continuing this way we obtain (not in a finitely many steps) the required 'broom digraph' D.

The obtained digraph D has indegree 2 and outdegree n. Moreover, it is easy to see that it is highly arc-transitive and has one end. Furthermore, let $v \in V(D)$. Then the sets $R_k^-(v)$ coincide with the sets $R_k^-(v)$ in the corresponding copy of G. Since T has indegree equal to 1 and outdegree equal to n, it is thus obvious that $|R_k^-(v)| = n^k$. Then, by Proposition 3.7, both conditions of column 4 in Table 1 are satisfied. △

Observe that Table 1 implies that in two-ended digraphs with property $\mathbf{Z}$ both of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ are finite. By Corollary 3.5 there exists a minimal integer $k \geq 1$ for which $R^+ = R_k^+ = R_k^- = R^-$. However, as the next infinite family of examples (which are even highly arc-transitive) shows, there is no universal upper bound on such k.

Example 4.11. Let $\bar{\mathbf{Z}}$ be the multidigraph obtained from the two-way infinite line $\mathbf{Z}$ by replacing each edge with a pair of parallel edges (both having the same direction as the original edge). For each $k \geq 2$ we construct a certain regular elementary abelian $\mathbb{Z}_2^k$-cover D_k of $\bar{\mathbf{Z}}$.

To this end we fix some notation. Let $N_k = \{0, 1, 2, 3, \ldots, 2^k - 1\} \subset \mathbb{Z}$. Let $\vec{\mathrm{Bin}}\colon N_k \to \mathbb{Z}_2^k$ be the function which assigns to each $j \in N_k$ the vector $\vec{\mathrm{Bin}}(j) = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}_2^k$ in such a way that $j = \sum_{i=1}^{k} j_i\, 2^{k-i}$ (that is, $\vec{\mathrm{Bin}}(j)$ corresponds to the binary expansion of j in a natural way).

Let now the $\mathbb{Z}_2^k$-cover D_k arise from the following ordinary voltage assignment on $\bar{\mathbf{Z}}$. For each vertex j of $\bar{\mathbf{Z}}$ we assign to the two edges connecting j to $j + 1$ the voltages $\vec{\mathrm{Bin}}(0)$ and $\vec{\mathrm{Bin}}(2^{t+1} - 1)$, where $j \equiv t \pmod{k}$. For example, if $k = 3$, then the edges connecting 0 to 1 have voltages $(0, 0, 0)$ and $(0, 0, 1)$, the edges connecting 1 to 2 have voltages $(0, 0, 0)$ and $(0, 1, 1)$, and the edges connecting 2 to 3 have voltages $(0, 0, 0)$ and $(1, 1, 1)$. Note that the subdigraphs induced on the union of fibres $\mathrm{Fib}(jk), \mathrm{Fib}(jk + 1), \ldots, \mathrm{Fib}((j + 1)k - 1)$, where $j \in \mathbb{Z}$, are all pairwise isomorphic and in fact one copy of it repeats periodically.

We now show that the digraph D_k is a connected infinite locally finite transitive two-ended digraph with property $\mathbf{Z}$ for which $R^+ = R_k^+ = R_k^- = R^-$, but $R_{k-1}^+ \neq R_k^+$ and $R_{k-1}^- \neq R_k^-$.

It is clear that D_k is a connected infinite locally finite two-ended digraph with property $\mathbf{Z}$. Moreover, the equivalence classes of $R^+ = R_k^+ = R_k^- = R^-$ coincide with the fibres of the projection onto $\mathbf{Z}$, whereas R_{k-1}^+ and R_{k-1}^- both have two equivalence classes in each fibre (and the classes of R_{k-1}^+ do not coincide with those of R_{k-1}^-).

We now show that D_k is transitive. In fact, D_k is even highly arc-transitive. First note that the nonzero voltages are precisely of the form $b_i = \vec{\mathrm{Bin}}(2^i - 1)$ for all $i \in \{1, 2, \dots, k\}$ and that they form a basis of the vector space $\mathbb{Z}_2^k$ over $\mathbb{Z}_2$. Observe that the net voltage of any walk in $\bar{\mathbf{Z}}$ is a linear combination over $\mathbb{Z}_2$ of the base vectors b_i. Moreover, for any closed walk and for any pair of parallel edges along this walk, the parity of the number of traversals of these two edges is the same. Hence a closed walk has net voltage 0 if and only if each edge in the walk is traversed an even number of times. This property is clearly preserved by any automorphism of $\bar{\mathbf{Z}}$. Consequently, again using the basic lifting lemma [8], the full automorphism group of $\bar{\mathbf{Z}}$ lifts. This shows that D_k is highly arc-transitive. △

4.3. Growth

We conclude the paper with a result connecting properties of the relations R_k^+ and R_k^- and the growth of a digraph.

Theorem 4.12. *Let D be a connected infinite locally finite transitive digraph. If at least one of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite, then D has exponential growth.*

Proof. Without loss of generality we assume that $(R_k^+)_{k\in\mathbb{Z}^+}$ is infinite. Proposition 4.3 implies that D has property $\mathbf{Z}$. Let $\psi : D \to \mathbf{Z}$ be an epimorphism and let $F_j = \psi^{-1}(j)$ denote its fibres. Clearly, each equivalence class of any R_k^+ is entirely contained in some fibre F_j. Also, since the sequence $(R_k^+)_{k\in\mathbb{Z}^+}$ is infinite, each F_j contains infinitely many equivalence classes with respect to any R_k^+, and moreover, these classes are nested. In order to show that D has exponential growth we inductively construct – as a subdigraph of D – a binary tree of arbitrarily large height.

For each $u \in F_0$ and each $k \geq 1$ let C_u^k denote the subdigraph of D induced by all walks $W \in R_k^+[u', u'']$, where $u', u'' \in R_k^+(u)$. Note that the subdigraphs C_u^k only contain vertices of fibres F_0 through F_k. Moreover, as the sequence $(R_k^+)_{k\in\mathbb{Z}^+}$ is infinite, each C_u^k is properly contained in C_u^{k+1}.

The binary trees of arbitrarily large height are now constructed inductively as follows. First observe that in each C_u^1 the vertices which are in F_1 have indegree at least 2 in C_u^1, and so each of them is the root of a binary tree of height 1 which is contained in some C_u^1. By transitivity, each vertex of F_1 is the root of a binary tree of height 1 which is a subdigraph of some C_u^1. Next, each C_u^2 contains at least two different subdigraphs from $\{C_{u'}^1 \mid u' \in F_0\}$, say $C_{u_1}^1$ and $C_{u_2}^1$. Thus there exist vertices $w_1 \in C_{u_1}^1 \cap F_1$, $w_2 \in C_{u_2}^1 \cap F_1$ and $v \in C_u^2 \cap F_2$ such that $(w_1, v), (w_2, v) \in E(D)$. It follows that v is a root of a binary tree of height 2 which is contained in C_u^2. By transitivity every vertex of F_2 is a root of a binary tree of height 2 which is contained in some C_u^2.

The induction step is now at hand. Each C_u^{k+1}, where $k \geq 1$, contains at least two different subdigraphs from $\{C_{u'}^k \mid u' \in F_0\}$, say $C_{u_1}^k$ and $C_{u_2}^k$. Thus there exist vertices $w_1 \in C_{u_1}^k \cap F_k$, $w_2 \in C_{u_2}^k \cap F_k$ and $v \in C_u^{k+1} \cap F_{k+1}$ such that $(w_1, v), (w_2, v) \in E(D)$. (Such vertices exist since, by assumption, $C^k_{u_1}$ and $C^k_{u_2}$ are both contained in C^{k+1}_u.) By induction hypothesis w_1 and w_2 are roots of binary trees of heights k which are disjoint, as $C^k_{u_1}$ and $C^k_{u_2}$ are different. It follows that v is a root of a binary tree of height $k+1$ which is contained in C^{k+1}_u. By transitivity every vertex of F_{k+1} is a root of a binary tree of height $k+1$ which is contained in some C^{k+1}_u.

By induction we have binary trees of arbitrarily large height in D, as required. □

We mention that Theorem 4.12 can also be deduced from a result of Trofimov [15]. But this deduction also needs several lines of proof, and therefore, to keep this paper self-contained, we gave the above elementary proof.

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